

$U(N)$ invariant dynamics for a simplified loop quantum gravity model

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Abstract. The implementation of the dynamics in Loop Quantum Gravity (LQG) is still an open problem. Here, we discuss a tentative dynamics for the simplest class of graphs in LQG: Two vertices linked with an arbitrary number of edges. We use the recently introduced $U(N)$ framework in order to construct $SU(2)$ invariant operators and define a global $U(N)$ symmetry that will select the homogeneous/isotropic states. Finally, we propose a Hamiltonian operator invariant under area-preserving deformations of the boundary surface and we identify possible connections of this model with Loop Quantum Cosmology.

1. Introduction

Loop quantum gravity (LQG) is now a well-established approach to quantum gravity [1]. It provides a non-perturbative mathematical formulation of the kinematical sector of the theory. The Hilbert space is generated by spin-networks: wave functions defined over oriented graphs whose edges are labeled by irreducible representations of the $SU(2)$ group, and with intertwiners ($SU(2)$ invariant tensors) on its vertices. Despite the several advances that have taken place in

this field, one of the main challenges faced by the theory is the implementation of the dynamics. Our goal is to focus on a specific model in order to propose a suitable Hamiltonian for it.

Rovelli and Vidotto introduced a simple model based on a graph with 2 vertices and 4 edges [2]. They found that this model leads to a physical framework very similar to Loop Quantum Cosmology (LQC). We generalize this model and implement the LQG dynamics on a graph with 2 vertices joined by an arbitrary number N of edges [3].

We use the recently developed $U(N)$ framework for $SU(2)$ -intertwiners [4, 5, 6] in order to study the Hilbert space of spin networks on the 2-vertex graph. The operators of the $U(N)$ formalism act on our Hilbert space and we identify a global $U(N)$ symmetry generated by operators acting on the coupled system of the two vertices which reduces the full space of arbitrary spin network states to a space of homogeneous/isotropic states. Finally, we introduce global $U(N)$ -invariant operators and use them to propose a $U(N)$ -invariant Hamiltonian operator.

2. The $U(N)$ framework

This framework was introduced in a series of papers [4, 5, 6]. Intertwiners with N legs are $SU(2)$ -invariant states in the tensor product of N (irreducible) representations of $SU(2)$. Then the basic tool used is the Schwinger representation of the $\mathfrak{su}(2)$ Lie algebra in terms of a pair of harmonic oscillators. Since we would like to describe the tensor product of N $SU(2)$ -representations, we will need N copies of the $\mathfrak{su}(2)$ -algebra and thus we consider N pairs of harmonic oscillators a_i, b_i with i running from 1 to N .

We can identify $SU(2)$ invariant operators acting on pairs of (possibly equal) legs i, j [4, 6]:

$$E_{ij} = a_i^\dagger a_j + b_i^\dagger b_j, \quad (E_{ij}^\dagger = E_{ji}), \quad F_{ij} = a_i b_j - a_j b_i, \quad (F_{ji} = -F_{ij}).$$

The operators E form a $\mathfrak{u}(N)$ -algebra

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj},$$

and with the operators F, F^\dagger form a closed algebra. This is why this formalism has been dubbed the $U(N)$ framework for LQG. Notice that the diagonal operators give the energy on each leg, $E_{ii} = E_i$. Then the value of the total energy $E \equiv \sum_i E_i$ gives twice the sum of all spins $2 \times \sum_i j_i$, i.e twice the total area around the vertex in the context of LQG.

The E_{ij} -operators change the energy/area carried by each leg, while still conserving the total energy, while the operators F_{ij} (resp. F_{ij}^\dagger) will decrease (resp. increase) the total area E by 2:

$$[E, E_{ij}] = 0, \quad [E, F_{ij}] = -2F_{ij}, \quad [E, F_{ij}^\dagger] = +2F_{ij}^\dagger.$$

This suggests to decompose the Hilbert space of N -valent intertwiners into subspaces of constant area:

$$\mathcal{H}_N = \bigoplus_{\{j_i\}} \text{Inv} \left[\bigotimes_{i=1}^N V^{j_i} \right] = \bigoplus_{J \in \mathbb{N}} \bigoplus_{\sum_i j_i = J} \text{Inv} \left[\bigotimes_{i=1}^N V^{j_i} \right] = \bigoplus_J \mathcal{H}_N^{(J)},$$

where V^{j_i} denotes the Hilbert space of the irreducible $SU(2)$ -representation of spin j_i , spanned by the states of the oscillators a_i, b_i with fixed total energy $E_i = 2j_i$. Besides, it was proven [5] that each subspace $\mathcal{H}_N^{(J)}$ of N -valent intertwiners with fixed total area J carries an irreducible representation of $U(N)$ generated by the E_{ij} operators.

Then the operators E_{ij} allow to navigate from state to state within each subspace $\mathcal{H}_N^{(J)}$. On the other hand, the operators F_{ij}, F_{ij}^\dagger allow to go from one subspace $\mathcal{H}_N^{(J)}$ to the next $\mathcal{H}_N^{(J \pm 1)}$, thus endowing the full space of N -valent intertwiners with a Fock space structure

with creation operators F_{ij}^\dagger and annihilation operators F_{ij} . It is worth to point out that the operators $E_{ij}, F_{ij}, F_{ij}^\dagger$ satisfy certain quadratic constraints [3] that look a lot like constraints on the multiplication of two matrices, one of them hermitian and the other antisymmetric [7].

3. The 2 vertex model

We consider the simplest class of non-trivial graphs for spin network states in Loop Quantum Gravity: a graph with two vertices linked by N edges, as shown in fig.1. This is a generalization of the simplest model (2 vertices and 4 links) introduced by Rovelli and Vidotto in [2] and which was shown to be related to models of quantum cosmology [8].

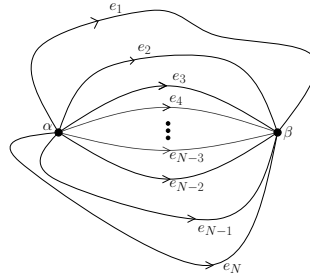


Figure 1. The 2-vertex graph with vertices α and β and the N edges linking them.

There are matching conditions [5] imposing that each edge carries a unique $SU(2)$ representation, thus the spin on that edge must be the same as seen from α than from β i.e. $j_i^\alpha = j_i^\beta$. This translates into the fact that the oscillator energy for α on the leg i must be equal to the energy for β on its i -th leg:

$$\mathcal{E}_i \equiv E_i^{(\alpha)} - E_i^{(\beta)} = 0.$$

We can see that the matching conditions \mathcal{E}_k generate a $U(1)^N$ symmetry, so they are part of a larger $U(N)$ symmetry algebra. Indeed, it is possible to introduce the more general operators:

$$\mathcal{E}_{ij} \equiv E_{ij}^{(\alpha)} - E_{ji}^{(\beta)} = E_{ij}^{(\alpha)} - (E_{ij}^{(\beta)})^\dagger.$$

that form a $U(N)$ algebra and that reduce to the matching conditions in the case $i = j$.

Now, we can show [3] that by imposing the global $U(N)$ -invariance just defined on our 2-vertex system, we obtain a single state $|J\rangle$ for every total boundary area J . Thus, the $U(N)$ invariance is restricting our system to states which are not sensitive to area-preserving deformations of the boundary between α and β . They are isotropic in the sense that all directions (i.e. all edges) are equivalent and the state only depends on the total boundary area, and they are homogeneous in the sense that the quantum state is the same at every point of space, i.e. at α and β .

In the following, we will restrict our study to this global $U(N)$ invariant Hilbert space and we will propose a dynamics for this system. Studying the structure of the $U(N)$ invariant operators acting on this space, we propose the simplest and most natural ansatz for a Hamiltonian operator:

$$H \equiv \lambda \sum_{ij} E_{ij}^{(\alpha)} E_{ij}^{(\beta)} + \left(\sigma \sum_{ij} F_{ij}^{(\alpha)} F_{ij}^{(\beta)} + \bar{\sigma} \sum_{ij} F_{ij}^{\alpha\dagger} F_{ij}^{\beta\dagger} \right),$$

where the coupling λ is real while σ can be complex a priori, so that the operator H is Hermitian. In fact, this is the most general $U(N)$ invariant Hamiltonian (allowing only elementary changes in the total area), up to a renormalization by a E -dependent factor.

The action of this Hamiltonian over a state $|J\rangle$ is known, and we can also study the spectral properties of this operator. It turns out that it shares several mathematical analogies with the evolution operator used in Loop Quantum Cosmology [9, 10]. From this perspective, the 2-vertex model has a natural cosmological interpretation and exhibits a “big bounce” behavior avoiding the big bang singularity as in LQC.

4. Conclusions

The $U(N)$ framework recently introduced in [4, 5, 6] represents a new and refreshing way to tackle several issues in the Loop Quantum Gravity framework.

In this work we have described a very simple model for LQG, based on the 2-vertex graph described above. We have the space of intertwiners for each vertex, plus some matching conditions coupling the two intertwiners by imposing that each edge i of the graph carries a unique spin label j_i as seen from both vertices. These matching conditions form the Cartan subalgebra of a larger “global” $\mathfrak{u}(N)$ algebra acting on both vertices, and we have seen that the invariance under this global $U(N)$ symmetry implies the restriction to isotropic and homogeneous states $|J\rangle$ at the quantum level. This provides our $U(N)$ symmetry with a very concrete physical interpretation. Indeed it could be viewed as a key step towards a full understanding between the general Loop Quantum Gravity (LQG) framework and the symmetry-reduced Loop Quantum Cosmology (LQC).

We have focused on the global $U(N)$ invariant space of spin-network states. Using suitable $U(N)$ invariant operators we have proposed a dynamics for this system. The defined $U(N)$ invariant Hamiltonian allows a direct comparison with the Hamiltonian constraints used in LQC. This fact reinforces the possible connection between this kind of models and the Loop Quantum Cosmology framework.

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